Chapter 3.1 Exponential Functions

→ Exponential function f with base b.

$$f(x) = b^x \text{ or } y = b^x$$

where b is a positive constant other than 1 ($b > 0$ and $b \neq 1$)

Yes

$y = 2^x$ $y = 10^x$ $y = 3^{x+1}$ $y = \left(\frac{1}{2}\right)^{x-1}$

No

$y = x^2$ — variable

$y = 1^x$ — always = 1 so constant

$y = (-1)^x$ — neg.

$y = x^x$

① Evaluate exponential functions

Ex 1] Find the average amount spent, to the nearest $, after 3 hrs at mall. Does this rounded function value under/over estimate the amount shown here → $149 ← from bar chart

$$f(x) = 42.2(1.56)^x$$

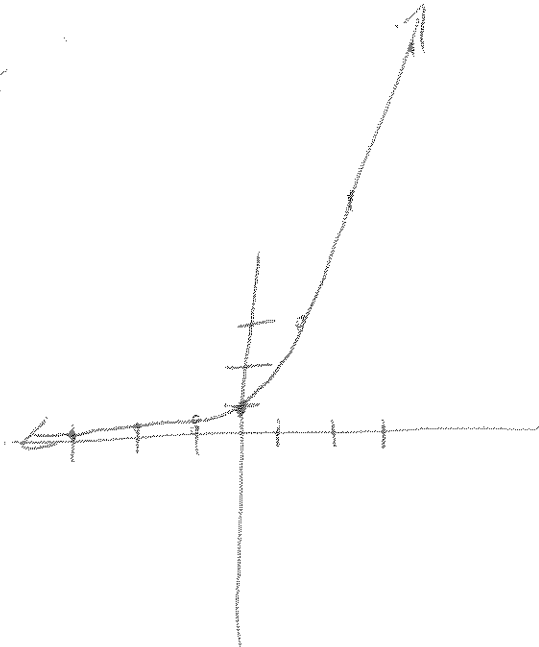
So, $f(3) = 42.2(1.56)^3 \approx 160.20876 \approx 160$

∴ So $160 is an overestimate of $11

② Graph Exponential Functions

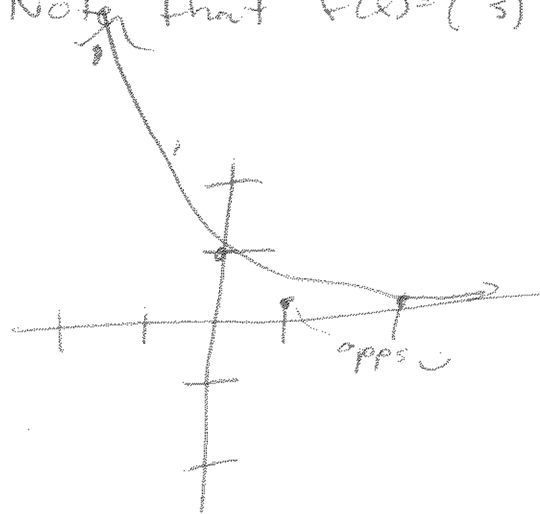
Ex Graph $f(x) = 3^x$

x	$f(x) = 3^x$
-3	$3^{-3} = \frac{1}{27}$
-2	$3^{-2} = 1/9$
-1	$3^{-1} = 1/3$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$
3	$3^3 = 27$



Ex Graph: $f(x) = \left(\frac{1}{3}\right)^x$. Note that $f(x) = \left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x}$

x	$f(x) = 3^{-x}$
-2	$3^{-(-2)} = 3^2 = 9$
-1	$3^{-1(-1)} = 3^1 = 3$
0	$3^0 = 1$
1	$3^{-1} = 1/3$
2	$3^{-2} = 1/9$

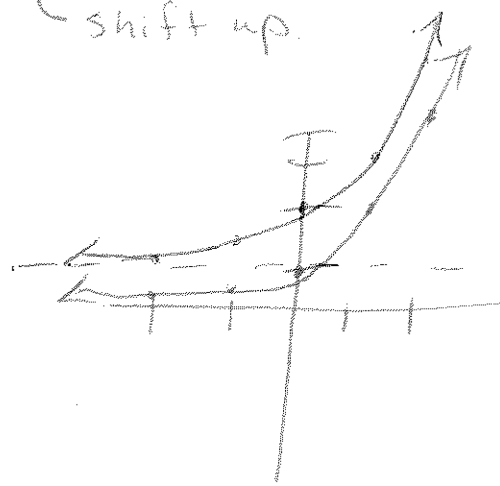


Note: look at table 3.1 on pg. 392

Ex | Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^x + 1$ | C 3.1

shift up

x	$f(x) = 2^x$	$f(x) = 2^x + 1$
-2	$2^{-2} = 1/4$	$1 + 1/4 = 5/4$
-1	$2^{-1} = 1/2$	$1 + 1/2 = 1.5$
0	$2^0 = 1$	$1 + 1 = 2$
1	$2^1 = 2$	$1 + 2 = 3$
2	$2^2 = 4$	$1 + 4 = 5$

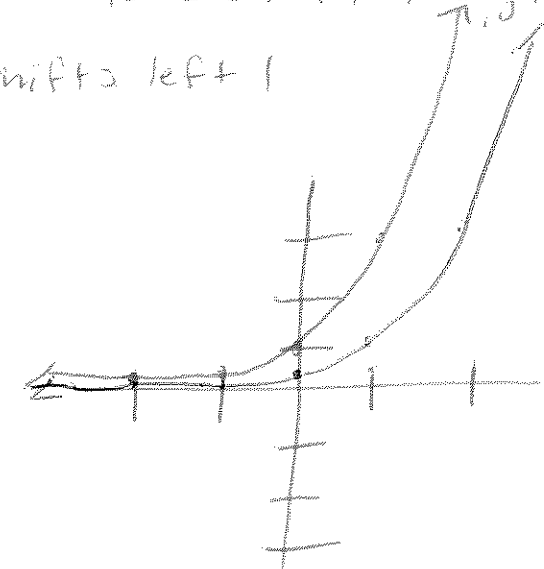


HA@1

Ex | Use the graph of $f(x) = 3^x$ to obtain the graph of $g(x) = 3^{x-1}$

shifts left 1

x	$f(x) = 3^x$	$g(x) = 3^{x-1}$
-2	$3^{-2} = 1/9$	$3^{-2-1} = 3^{-3} = 1/27$
-1	$3^{-1} = 1/3$	$3^{-1-1} = 3^{-2} = 1/9$
0	$3^0 = 1$	$3^{0-1} = 3^{-1} = 1/3$
1	$3^1 = 3$	$3^{1-1} = 3^0 = 1$
2	$3^2 = 9$	$3^{2-1} = 3^1 = 3$



③ Evaluate functions with base e.

Ex | The exponential func. $f(x) = 1066e^{0.042x}$ models the gray wolf population of the western Great Lakes, $f(x)$, x years after 1978. What is population in 2012

$x = 34$ yrs since 1978.

$f(34) = 1066e^{0.042(34)} = 4446$ wolf.

(4) Use compound interest Formulas

→ For n compounding per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

A = Final balance

P = Principal

r = rate (as decimal)

n = # of times compounded/yr.

t = years

→ For continuous compounding $A = Pe^{rt}$

Ex 7) A sum of \$10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to

a.) Quarterly compounding

$n = 4$ (since quarterly)

$A = 10000\left(1+\frac{0.08}{4}\right)^{4\cdot 5}$

$\underline{A = \$14,859.47}$

b.) Continuous compounding

$A = Pe^{rt}$

$A = 10000\,e^{0.08\cdot 5}$

$\underline{A = 14,918.24}$

Chapter 3.2 Logarithmic Functions

→ inverse of exponential

① Change from logarithmic to exponential form.

Ex a.) $3 = \log_7 x$ b.) $2 = \log_b 25$ c.) $\log_4 26 = y$

$7^3 = x$ $b^2 = 25$ $4^y = 26$

② Change from exponential to logarithmic form.

Ex a.) $2^5 = x$ b.) $b^3 = 27$ c.) $e^y = 33$

$5 = \log_2 x$ $3 = \log_b 27$ $y = \log_e 33$

③ Evaluate logarithms

Ex Evaluate ⟹ common log = log $\log_{10}$

a.) $\log_{10} 100$ b.) $\log_5 \frac{1}{125}$ c.) $\log_{36} 6$ d.) $\log_3 \sqrt[7]{3}$

$10^x = 100$ $5^x = \frac{1}{125}$ $36^x = 6$ $3^x = \sqrt[7]{3}$

$x = 2$ $x = -3$ $x = 1/2$ $x = 1/7$

④ Use basic logarithmic properties

$\log_b b = 1$

$\log_b 1 = 0$

Ex $\log_9 9 = 1$

$\log_8 1 = 0$

Stop here!

$\log_b b^x = x$

$b^{\log_b x} = x$

Ex a.) $\log_7 7^8 = 8$

b.) $3^{\log_3 17} = 17$

⑤ Graph logarithmic Functions

$F(x) = a \log_b (x-h) + k$

a: if $a > 1$ vertical stretch
$\cancel{1}$ $a < 1$ vertical comp.

h: opp h
vertical Asymptote

k: $+k\uparrow$, $-k\downarrow$

- Graph asymptote (opp h value)
- Go up/down k value, Go right 1 from asymptote.
- Go back to asymptote, go up/down a value,
- Go right base value.
- Graph.

Ex Graph and find the domain $f(x) = \log_4 (x-5)$

D: $(5, \infty)$
R: $(-\infty, \infty)$

(8) use natural logs

$$LN = \ln = \log_e$$

<u>General Properties</u>	<u>Natural logs</u>
1.) $\log_b 1 = 0$	$\ln 1 = 0$
2.) $\log_b b = 1$	$\ln e = 1$
3.) $\log_b b^x = x$	$\ln e^x = x$
4.) $b^{\log_b x} = x$	$e^{\ln x} = x$

Ex | Find the domain

a.) $F(x) = \log_4 (x-5)$

$x - 5 > 0$

$x > 5$ $\quad D: (5, \infty)$

b.) ~~ln~~ $F(x) = \ln(4-x)$

$4 - x > 0$

$x < 4$ $\quad D: (-\infty, 4)$

c.) $h(x) = \ln x^2$

$\sqrt{x^2} \neq \sqrt{0}$

$x > 0$ and $x < 0$

$D: (-\infty, 0) \cup (0, \infty)$

Chapter 3.3 Properties of logarithms

– log is inverse of exp over $y=x$
switch x, y

① use the product rule

– ~~talk~~ about $x^2 \cdot x^3 = x^{2+3}$

$$\log_b(MN) = \log_b M + \log_b N$$

Ex) Use the product rule to expand.

a.) $\log_6(7 \cdot 11)$

$\log_6 7 + \log_6 11$

b.) $\log(100x)$

$\log 100 + \log x$

$2 + \log x$

② Use the quotient rule

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Ex) Use the quotient rule to expand

a.) $\log_8\left(\frac{23}{x}\right)$

$\log_8 23 - \log_8 x$

b) $\ln\left(\frac{e^5}{11}\right)$

$\ln e^5 - \ln 11$

$5 - \ln 11$

③ Use the power rule

$$\log_b M^p = p \log_b M$$

Ex) Use power rule to expand

a.) $\log_6 3^9$

$9 \log_6 3$

b) $\ln \sqrt[3]{x}$

$\ln x^{1/3}$

$\frac{1}{3} \ln x$

c.) $\log(x+4)^2$

$2 \log(x+4)$

④ Expand logarithmic expressions

a.) $\log_6(x^4\sqrt[3]{y})$

$\log_6 x^4 + \log_6 y^{1/3}$

$4\log_6 x + \frac{1}{3}\log_6 y$

b.) $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

$\log_5 x^{1/2} - \log_5 25y^3$

$\log_5 x^{1/2} - \log_5 5^2 + \log_5 y^3$

$\frac{1}{2}\log_5 x - 2\log_5 5 + 3\log_5 y$

$\frac{1}{2}\log_5 x - 2 + 3\log_5 y$

⑤ Condense logarithmic expressions

$\boxed{Ex}$ a.) $\log 25 + \log 4$

$\log(25 \cdot 4)$

$\log(100)$

$\log 10^2 = 2$

b.) $\log(7x+6) - \log x$

$\log \frac{7x+6}{x}$

c.) $2\ln x + \frac{1}{3}\ln(x+5)$

$\ln x^2 + \ln(x+5)^{1/3}$

$\ln x^2(x+5)^{1/3}$

$\ln x^2\sqrt[3]{x+5}$

d.) $2\log(x-3) - \log x$

$\log(x-3)^2 - \log x$

$= \log\frac{(x-3)^2}{x}$

e.) $\frac{1}{4}\log_6 x - 2\log_6 5 - 10\log_6 y$

$\log_6 x^{1/4} - \log_6 5^2 - \log_6 y^{10}$

$\log_6 x^{1/4} - (\log_6 25 + \log_6 y^{10})$

$\log_6 x^{1/4} - (\log_6 25y^{10})$

$\log_6 \frac{x^{1/4}}{25y^{10}} = \log_6 \frac{\sqrt[4]{x}}{25y^{10}}$

⑥ use the Change-of-base property

$$\log_b M = \frac{\log_a M}{\log_a b}$$

Common and Natural logs

$$\log_b M = \frac{\log M}{\log b} \qquad \log_b M = \frac{\ln M}{\ln b}$$

Ex) Use common logs for $\log_7 2506$

$$\log_7 2506 = \frac{\log 2506}{\log 7} = 4.02$$

plug into calc

Ex) Use natural logs to evaluate $\log_7 2506$

$$\log_7 2506 = \frac{\ln 2506}{\ln 7} = 4.02$$

Chapter 3.4 Exponential and Logarithmic Equations

① Use like bases to solve exponential equations

-> Get bases =
-> set exponents =
-> solve

Ex Solve

a.) $5^{3x-6} = 125$

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$3x = 9$

$x = 3$

b.) $8^{x+2} = 4^{x-3}$

$2^{3(x+2)} = 2^{2(x-3)}$

$3x + 6 = 2x - 6$

$x = -12$

② Use log to solve exponential eqs.

-> Isolate exponential
-> Use ln for everything except base 10.
-> Simplify
-> Solve

Ex Solve

a.) $5^x = 134$

$x \ln 5 = \ln 135$

$x = \dfrac{\ln 135}{\ln 5}$

Solution set is $\left\{\dfrac{\ln 134}{\ln 5}\right\} \approx 3.04$

b.) $10^x = 8000$

$x \log 10 = \log 8000$

$x = \log 8000 \approx 3.90$

c.) Solve

$7e^{2x} - 5 = 58$

$\dfrac{7e^{2x}}{7} = \dfrac{63}{7}$

$\ln e^{2x} = \ln 9$

$\dfrac{2x}{2} = \dfrac{\ln 9}{2}$

$x = \dfrac{\ln 9}{2} \approx 1.10$

Solution set $\left\{\dfrac{\ln 9}{2}\right\} \approx 1.10$

Ex Solve $3^{2x-1} = 7^{x+1}$

$(2x-1)\ln 3 = (x+1)\ln 7$

$2x\ln 3 - \ln 3 = x\ln 7 + \ln 7$

$2x\ln 3 - x\ln 7 = \ln 3 + \ln 7$

$\frac{x(2\ln 3 - \ln 7)}{2\ln 3 - \ln 7} = \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7}$

$x = 12.11$

Ex Solve $e^{2x} - 8e^{x} + 7 = 0$

$(e^{x}-1)(e^{x}-7) = 0$

$e^{x} - 1 = 0 \qquad e^{x} - 7 = 0$

$e^{x} = 1 \qquad e^{x} = 7$

$\ln e^{x} = \ln 1 \qquad \ln e^{x} = \ln 7$

$x = \ln 1 \qquad x = \ln 7$

$x = 0$

$\{0, \ln 7\}$

$0, 1.95$

③ Use the definition of a logarithm to solve logarithmic Eq.

Ex a.) $\log_2(x-4) = 3$

$2^3 = x - 4$

$8 = x - 4$

$x = 12$

Check

$\log_2(12-4) = 3$

$\log_2 8 = 3$

$2^3 = 8$ ✓

b.) $\frac{4\ln(3x)}{4} = \frac{8}{4}$

$\ln(3x) = 2$

$e^{\ln 3x} = e^2$

$3x = e^2$

$x = \frac{e^2}{3} \approx 2.46$

Check

$4\ln\left(3\left(\frac{e^2}{3}\right)\right) = 8$

$4\ln e^2 = 8$

$4 \cdot 2 = 8$

$8 = 8$ ✓

C 3.4

c.) $\log x + \log(x-3) = 1$

$\log x(x-3) = 1$

$10^1 = x(x-3)$

$10 = x^2 - 3x$
$-10 \qquad -10$

$0 = x^2 - 3x - 10$

$0 = (x-5)(x+2)$

$x = 5, -2$

Check

$x = 5$

$\log 5 + \log 2 = 1$

$\log(5 \cdot 2) = 1$

$\log 10 = 1$

$1 = 1$ ✓

$x = -2$

$\log -2 + \log -5 = 1$

Negative # do not have logs

So solution set is $\{5\}$

④ Use the one-to-one property of logs to solve logs

Ex] Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right)$

$(x+1) \cdot x - 3 = \frac{7x-23}{x+1} \cdot (x+1)$

$x^2 + x - 3x - 3 = 7x - 23$

$x^2 - 2x - 3 = 7x - 23$
$-7x + 23 \quad -7x + 23$

$x^2 - 9x + 20 = 0$

$(x-4)(x-5) = 0$

$x = 4, 5$

Check

$\ln(4-3) = \ln\left(\frac{7(4)-23}{4+1}\right)$

$\ln(1) = \ln\left(\frac{28-23}{5}\right)$

$\ln(1) = \ln\left(\frac{5}{5}\right)$

$\ln(1) = \ln(1)$ ✓

$\ln(5-3) = \ln\left(\frac{7(5)-23}{5+1}\right)$

$\ln 2 = \ln\left(\frac{35-23}{6}\right)$

$\ln 2 = \ln\left(\frac{12}{6}\right)$

$\ln 2 = \ln 2$ ✓

Chapter 3.5 Exponential Growth & Decay; Modeling Data

I) Model exponential growth & decay.

$f(t) = A_0 e^{kt}$ or $A = A_0 e^{kt}$

A_0 is original amount

t = time

A of $f(t)$ = final amount

k is the growth/decay constant

If $k > 0$ the function models a growth function

If $k < 0$ " " Decay function

Ex 1) In 1990, the population of Africa was 643 million and by 2006 it had grown to 906 million.

a.) Use the exponential growth model $A = A_0 e^{kt}$ in which t is the number of years after 1990 to find the exp

b.) By which year will Africa's pop. reach 2000 million, or 2 billion?

a.) $A_0 = \cancel{8}643$

$t = 16$ $A = 906$

$\frac{906}{643} = \frac{643 e^{k(16)}}{643}$

$\ln \frac{906}{643} = \ln e^{16k}$

$\frac{16k}{16} = \frac{\ln\left(\frac{906}{643}\right)}{16}$

$k = \frac{\ln\left(\frac{906}{643}\right)}{16} = 0.021$

So the function is

$A = A_0 e^{0.021t}$

Ex 1 cont'd

b.) $A = 643e^{0.021t}$

$\frac{2000}{643} = \frac{643e^{0.021t}}{643}$

$\ln \frac{2000}{643} = \ln e^{0.021t}$

$\frac{\ln\left(\frac{2000}{643}\right)}{0.021} = \frac{0.021t}{0.021}$

$t \approx 54$ ∴ $1990 + 54 =$ In 2044 Africa's pop will reach 2 billion

Ex 2 Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmospheric nuclear tests, we all have a measurable amount of strontium-90 in our bones.

a.) The half-life of strontium-90 is 28 yrs. Find the exponential decay model for strontium-90.

b.) Suppose that a nuclear accident occurs & releases 60 grams of strontium-90 into the atmosphere. How long will it take to decay to 10 grams

a.) $A = A_0 e^{kt}$

$\ln \frac{1}{2} = \ln e^{k(28)}$

$\frac{\ln\left(\frac{1}{2}\right)}{28} = \frac{28k}{28}$

$k \approx -0.0248$

∴ $A = A_0 e^{-0.0248t}$

Ex 2 cont'd

b.) $A = A_0 e^{-0.0248t}$

$$\frac{10}{60} = \frac{60e^{-0.0248t}}{60}$$

$$\ln\frac{1}{6} = \ln e^{-0.0248t}$$

$$\frac{\ln\frac{1}{6}}{-0.0248} = \frac{-0.0248t}{-0.0248}$$

$t \approx 72$ years

② Use logistic growth models

$$f(t) = \frac{c}{1+ae^{-bt}} \quad \text{or} \quad A = \frac{c}{1+ae^{-bt}}$$

a, b, c are constants, with $c > 0$ & $b > 0$.

Ex) In a learning theory project, psychologists discovered that $f(t) = \frac{0.8}{1+e^{-0.2t}}$ after t learning trials

a.) Find the proportion of correct responses prior to learning trials taking place.

b.) Find the proportion of correct responses after 10 learning trials

c.) What is the limiting size of $f(t)$, the proportion of correct responses, as continued learning trials take place?

a.) $t = 0$ $\quad f(0) = \frac{0.8}{1+e^{-0.2(0)}} = 0.4$

Ex cont'd

b.) $t=10$ $\quad f(10)=\dfrac{0.8}{1+e^{-0.2(10)}} \approx 0.7$

c.) In a logistic model, c, is the limiting size so $\therefore$ 0.8 is the limiting size.

③ Use Newton's Law of Cooling

$$T = C + (T_0 - C)e^{kt}$$

C = Ambient Temp
T_0 = Initial Temp
k = negative constant
t = time
T = Final Temp.

Ex) An object is heated to 100°C. It is left to cool in a room that has a temp of 30°C. After 5 minutes, the temp of the object is 80°C.

a.) Use Newton's Law of Cooling to find a model for the temp. of the object, T, after t minutes.
b.) what is the temp of the object after 20 minutes?
c.) when will the temp. of the object be 35°C.

a.) $80 = 30 + (100-30)e^{k5}$

$\dfrac{50}{70} = \dfrac{70e^{5k}}{70}$

$\ln \dfrac{5}{7} = \ln e^{5k}$

$\dfrac{5k}{5} = \dfrac{\ln\left(\frac{5}{7}\right)}{5}$

$k = -0.0673$

$T = 30 + 70e^{-0.0673t}$

Ex cont'd

b.) $T = 30 + 70e^{-0.0673(20)} \approx 48°$

c.) $35 = 30 + 70e^{-0.0673t}$

$5 = 70e^{-0.0673t}$

$\ln \frac{5}{70} = \ln e^{-0.0673t}$

$$\frac{\ln\left(\frac{5}{70}\right)}{-0.0673} = \frac{-0.0673t}{-0.0673}$$

$t \approx 39$ min.